Small-angle approximation in the description of radiative collective effects within an ultrarelativistic electron bunch

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The problem of the evaluation of radiative collective effects accompanying accelerated motion of a short ultrarelativistic electron bunch in vacuum is considered within the framework of the small-angle approximation; second order expansion in the transverse velocity of electrons is performed in order to obtain an analytical expression for energy spread within the bunch. Comparison with earlier results by other authors shows good agreement.

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I. INTRODUCTION

Very short, high-charge bunches of electrons will be produced by particle accelerators of the next generation. Bunch compression chicanes are expected to be often used in order to provide very high-peak-current beams for x-ray self-amplified spontaneous emission (SASE)-free-electron lasers. Electron bunches of this kind could also be interesting for the development of high-brightness cherenkov and transition radiation sources. However, their production and utilization may prove difficult due to radiative collective effects.

A well-known example of such a collective effect is the enhancement of low-frequency photon emission from a short relativistic electron bunch moving along a circular trajectory [in the rest of the paper, we will refer to it as the steady-state coherent synchrotron radiation (CSR)]: the number of photons per unit frequency interval increases dramatically in the part of the spectrum where the photon wavelength becomes comparable with the size of the bunch. In this frequency range, electromagnetic waves emitted by individual particles have small phase differences. As a result, they add up coherently, thus leading to a quadratic dependence of the intensity of radiation on the number of electrons in the bunch [1]. This number is typically 10^8–10^{10}, which explains the high magnitude of the effect.

Similar effects are observed when an electron bunch passes bending magnets, magnetic chicanes, and other beam-optic elements. In all such cases, signal retardation is the crucial feature. In contrast to the above case of steady-state CSR, in any beam-optic system transient collective phenomena also take place. Their study has been a matter of active theoretical, numerical, and experimental research in the past few years.

The problem of a one-dimensional (1D) electron bunch entering a circular path from a straight path in vacuum has been carefully studied in [2]. The total energy loss due to collective effects as well as the final energy spread have been examined in several limiting cases that are of relevance for practical applications. The influence of shielding in a similar situation has been addressed in [3] and [4]. The presence of conducting walls has been shown to reduce the strength of radiative collective self-interactions in the bunch. Extensive numerical simulations were performed in [5–7]; a comparison with experimental results can be found in [8]. Measurements and computations are in reasonable agreement.

In the present article we consider the problem of radiative collective interactions within a short electron bunch following its trajectory in vacuum without shielding. The unusual feature of our consideration is that we consistently apply the small-angle approximation, a natural technique for ultrarelativistic particles. This approach considerably reduces the effort necessary for the treatment of an arbitrary trajectory; on the other hand, it somewhat restricts the class of allowed trajectories. From a conceptual point of view, it can account more easily for the effects of finite transverse extent of the bunch, because it does not make use of polar coordinates as other techniques do [3,9], thus getting rid of any extra term arising from the Jacobian of the transformation. Eventually, this route is expected to lead to an efficient computational tool for the design of magnetic systems for high-peak-current electron bunches.

The paper is organized as follows. In Sec. II the geometry of the problem is described and the small-angle approximation is introduced. Section III is devoted to the case of retarded interaction between two individual electrons. In Sec. IV our consideration is extended to the case of a test particle interacting with the whole bunch, and the results are compared with those by other authors. Finally, Sec. V contains conclusions and speculations.

II. THE SMALL-ANGLE APPROXIMATION

Following other authors [2,4,10,11], we will consider the bunch as a ‘‘rigid,’’ 1D, charged object with a given linear charge density distribution. We define a Cartesian reference frame (x,y,z) as shown in Fig. 1, where the z axis coincides with the direction of the initial velocity.

Our consideration makes use of the small-angle approximation. That is, we assume that the bunch energy is high enough that the possible deflection of electrons from a straight line during their passage through the magnetic system is relatively small. To be specific, we will assume that, before and after the magnets ($z<0$ or $z>\overline{z}$), the bunch moves along a rectilinear path with constant velocity, while

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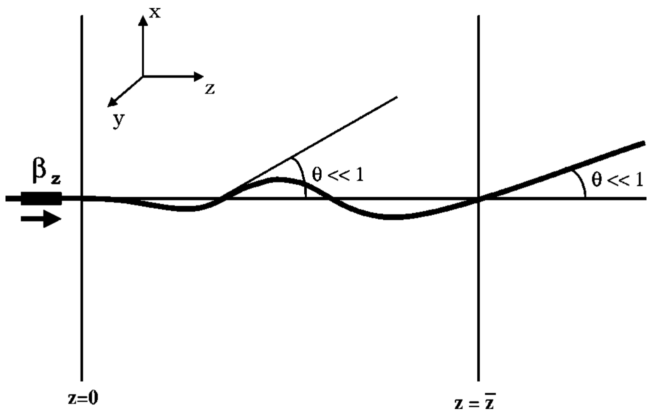

FIG. 1. Schematic of a particle trajectory in the small-angle approximation.

inside the magnetic system ($0<z<\bar{z}$) it follows a path subject to the only constraint that the angle θ formed by the velocity vector with the z axis is always small, i.e., $\theta \ll 1$. Note that θ can still be small or large as compared to the other small parameter of the problem, γ^{-1}, where $\gamma \gg 1$ is the Lorentz factor $\gamma = (\mathcal{E}/mc^2+1)$, $\mathcal{E}$ being the kinetic energy of the particles.

A natural assumption is $l_b(dv_{x,y}/dz) \ll v_{x,y}$, where l_b is the longitudinal extent of the bunch and $v_{x,y}(z)$ are the components of the transverse velocity of bunch electrons; in other words, we will consider a situation in which differences in transverse velocities of electrons are negligible. We will also assume zero initial energy spread in the bunch, and neglect any change of particle energy during the passage of the bunch through the magnetic system. This means that the trajectory of the bunch is predetermined by its initial energy and by the known configuration of external fields. The back influence of radiative effects on the motion of particles is therefore assumed to be negligible; of course, this assumption has to be verified *a posteriori*: in some cases of practical interest the energy change appears to be rather significant.

The two main objects we are going to deal with in the following calculations are the local particle velocity $\boldsymbol{v}$ and a unit vector $\hat{\boldsymbol{n}}$ connecting two points lying on the same trajectory. In the spirit of the small-angle approximation, one has to distinguish explicitly between their longitudinal and transverse components, assuming the latter to be small. Keeping first and second order terms and omitting all higher orders, one gets the following well-known expressions for the z components of the above vectors:

$$n_z \simeq 1 - \frac{1}{2}\boldsymbol{n}_\perp^2, \qquad (1)$$

$$v_z \simeq c\left(1-\frac{1}{2\gamma^2}\right) - \frac{\boldsymbol{v}_\perp^2}{2c}. \qquad (2)$$

Once the bunch trajectory is fixed, the problem of radiative collective effects within the bunch reduces to properly accounting for signal retardation in pairwise interactions between individual electrons. Let us consider a test particle inside the bunch. Its present velocity and its present position in the laboratory frame of reference will be denoted as $\boldsymbol{v}_0(t)$ and $\boldsymbol{r}_0(t)$, respectively. We are interested in its interaction with some other bunch particle—the source particle—whose present position will be denoted as $\boldsymbol{r}(t)$. Causality defines the well-known retardation condition between the two particles,

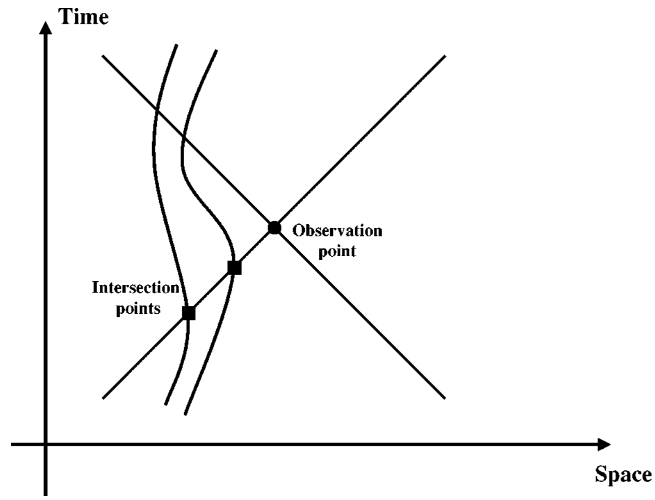

FIG. 2. Different particle world lines intersect the light cone of the observation event at different points in space-time.

$$|\boldsymbol{r}_0(t) - \boldsymbol{r}(t')| = c(t-t'), \qquad (3)$$

where $\boldsymbol{r}(t')$ denotes the retarded position of the source particle (t' being the so called retarded time), and $(t-t')$ is the time delay associated with signal propagation. Obviously, world lines of different source particles are intersecting the light cone of a certain event at different space-time points, as illustrated in Fig. 2.

The small-angle aproximation considerably simplifies the treatment of the above retardation condition. First, note that knowledge of the transverse velocity $\boldsymbol{v}_\perp$ as a function of time fully determines the position of a particle. Indeed, using Eq. (2) one gets, for the transverse ($\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$) and longitudinal coordinates of a particle

$$\boldsymbol{\rho}(t) = \int_0^t \boldsymbol{v}_\perp(\tau)\,d\tau, \qquad (4)$$

$$z(t) = z(0) + \int_0^t v_z(\tau)\,d\tau$$
$$\simeq z(0) + \int_0^t \left[c\left(1-\frac{1}{2\gamma^2}\right) - \frac{\boldsymbol{v}_\perp^2(\tau)}{2c}\right] d\tau. \qquad (5)$$

The transverse velocity, in its turn, is easily found once the configuration of external fields is defined, which makes this approach rather convenient.

Secondly, the positions of the test and of the source particle are related through

$$\boldsymbol{r}_0(t) = \boldsymbol{r}(t+\delta), \qquad (6)$$

because all particles in the bunch are assumed to follow the same trajectory. In the small-angle approximation the time difference δ is easily translated into the difference between z coordinates of the two particles:

$$\Delta z = z_0 - z \simeq c\,\delta. \tag{7}$$

It is worth mentioning that for $\delta > 0$ the position of the source particle is always *behind* that of the test particle. This is, in fact, the only case we are interested in. As has been argued in [2], interactions with particles that are *ahead* of the test particle do not contain a radiative part. Their contribution consists in trivial Coulomb repulsion, which has to be subtracted from the final expressions in order to get a nonsingular result (see the discussion of the Coulomb singularity in Sec. III). For this reason, in the following we will always assume $\Delta z > 0$.

Thirdly, it is convenient to switch from time retardation to a retardation condition expressed in z, which is possible since, in the small-angle approximation, t and z are uniquely mapped onto each other. The corresponding relation is easily found; namely, up to second order terms in the transverse velocity one gets

$$t - t' \simeq \frac{(z-z')}{c}\left(1 + \frac{1}{2\gamma^2}\right) + \frac{1}{2c}\int_{z'}^{z} d\zeta\, \boldsymbol{\beta}_\perp^2(\zeta), \tag{8}$$

where $\boldsymbol{\beta}_\perp$ is the usual notation for dimensionless velocity, $\boldsymbol{\beta}_\perp \equiv \boldsymbol{v}_\perp / c$. Using this, the retardation condition can be rewritten as

$$(z_0 - z')^2 + (\boldsymbol{\rho}_0 - \boldsymbol{\rho}')^2 \simeq \left[(z-z')\left(1 + \frac{1}{2\gamma^2}\right) + \frac{1}{2}\int_{z'}^{z} d\zeta\, \boldsymbol{\beta}_\perp^2(\zeta) \right]^2. \tag{9}$$

Rearranging terms, neglecting those of order $\Delta z/(z-z') \ll 1$, and taking into account Eq. (4), we find

$$\left[\int_{z'}^{z_0} d\zeta\, \boldsymbol{\beta}_\perp(\zeta) \right]^2 \simeq 2(z-z')\left[\frac{(z-z')}{2\gamma^2} - \Delta z + \frac{1}{2}\int_{z'}^{z} d\zeta\, \boldsymbol{\beta}_\perp^2(\zeta) \right]. \tag{10}$$

Finally, we represent the retardation condition in the small-angle approximation as

$$\frac{(z_0 - z')}{\gamma^2} + \int_{z'}^{z_0} d\zeta\, \boldsymbol{\beta}_\perp^2(\zeta) - \frac{1}{(z_0 - z')}\left(\int_{z'}^{z_0} d\zeta\, \boldsymbol{\beta}_\perp(\zeta) \right)^2 \simeq 2\Delta z. \tag{11}$$

III. LIENARD-WIECHERT FIELDS AND COULOMB SINGULARITY

The fields generated by a source particle at an observation point $\boldsymbol{r}_0(t)$ are given by the following expressions:

$$\boldsymbol{E}(\boldsymbol{r}_0, t) = \frac{e}{4\pi\varepsilon_0}\left\{ \frac{1}{\gamma^2}\, \frac{\hat{\boldsymbol{n}} - \boldsymbol{\beta}}{R^2(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3} + \frac{1}{c}\, \frac{\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{R(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3} \right\} \tag{12}$$

and

$$\boldsymbol{B}(\boldsymbol{r}_0, t) = \frac{1}{c}\, \hat{\boldsymbol{n}} \times \boldsymbol{E}, \tag{13}$$

where $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are, respectively, the dimensionless velocity and its time derivative at the retarded time t', R is the distance between the retarded position of the source particle and the observation point, and $\hat{\boldsymbol{n}}$ is a unit vector along the line connecting those two points.

Multiplying $e\boldsymbol{E}$ by the velocity of the test particle $\boldsymbol{v}_0$, one gets the change of the energy of the test particle due to its interacton with the source particle:

$$\left(\frac{d\mathcal{E}}{dt}\right) = e\boldsymbol{E}(\boldsymbol{r}_0, t)\cdot \boldsymbol{v}_0(t), \tag{14}$$

and hence

$$\left(\frac{d\mathcal{E}}{dt}\right) = \frac{e^2}{4\pi\varepsilon_0}\left[\frac{c}{\gamma^2}\, \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}_0 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}_0}{R^2(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3} + \frac{(\hat{\boldsymbol{n}}\cdot\dot{\boldsymbol{\beta}})(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}_0 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}_0) - (\boldsymbol{\beta}_0\cdot\dot{\boldsymbol{\beta}})(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})}{R(1 - \hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3} \right]. \tag{15}$$

As one can see, the above expression contains terms proportional to R^{-1} as well as those proportional to R^{-2}. The R^{-2} dependence resembles electrostatic interaction of two charges, which is why these are called the Coulomb terms. In contrast, the R^{-1} terms are called radiative ones.

As has been argued in [2], the Coulomb part is singular in the limit $R \to 0$ (that is, $\Delta z \to 0$). On the other hand, this large contribution has nothing to do with radiative effects, because it represents just trivial electrostatic repulsion of bunch electrons. Its singular behavior is connected with the infinitely small transverse size of the bunch that we use in our model problem. Following [2], we will cure the situation by subtracting from Eq. (15) its purely Coulomb counterpart corresponding to rectilinear motion of the same two particles with constant velocity:

$$\left(\frac{d\hat{\mathcal{E}}}{dt}\right) = \left(\frac{d\,\mathcal{E}}{dt}\right) - \frac{e^2 \beta c}{4\pi\varepsilon_0 \gamma^2 (\Delta z)^2}. \tag{16}$$

The resulting expression appears to be regular in the limit $\Delta z \to 0$. This regularized formula will be used in all following calculations.

In the small-angle approximation, one has to expand the above expressions up to second order terms in the transverse velocity. The following relations are quite helpful at this stage:

$$(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})\simeq\frac{1}{2}\left[\frac{1}{\gamma^2}+(\boldsymbol{\beta}_\perp-\boldsymbol{n}_\perp)^2\right], \tag{17}$$

$$(\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}_0-\boldsymbol{\beta}\cdot\boldsymbol{\beta}_0)\simeq\frac{1}{2}\left[\frac{1}{\gamma^2}+(\boldsymbol{\beta}_{\perp 0}-\boldsymbol{\beta}_\perp)^2-(\boldsymbol{\beta}_{\perp 0}-\boldsymbol{n}_\perp)^2\right], \tag{18}$$

$$(\boldsymbol{\beta}_0\cdot\dot{\boldsymbol{\beta}})\simeq(\boldsymbol{\beta}_{\perp 0}-\boldsymbol{\beta}_\perp)\cdot\dot{\boldsymbol{\beta}}_\perp, \tag{19}$$

where $\boldsymbol{n}_\perp$ is given by

$$\boldsymbol{n}_\perp=\frac{1}{(z_0-z')}\int_{z'}^{z_0}d\zeta\,\boldsymbol{\beta}_\perp(\zeta). \tag{20}$$

Using the above formulas and putting, with the same accuracy, $R\simeq(z_0-z')$, one gets

$$\left(\frac{d\hat{\mathcal{E}}}{dt}\right)\simeq\frac{e^2}{4\pi\varepsilon_0}\frac{2\gamma^2}{1+\gamma^2[\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z')]^2}\{[C]+[R]\}, \tag{21}$$

where $[C]$ and $[R]$ stand for the Coulomb and the radiative part, respectively,

$$\begin{aligned}[C]\equiv{}&\frac{2c}{(z_0-z')^2}\\&\times\left\{\frac{1-\gamma^2[\boldsymbol{\beta}_\perp(z_0)-\boldsymbol{n}_\perp]^2+\gamma^2[\boldsymbol{\beta}_\perp(z_0)-\boldsymbol{\beta}_\perp(z')]^2}{\{1+\gamma^2[\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z')]^2\}^2}\right.\\&\left.-\frac{1+\gamma^2[\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z')]^2}{\left[1-\gamma^2\boldsymbol{n}_\perp^2+\gamma^2(z_0-z')^{-1}\int_{z'}^{z_0}\boldsymbol{\beta}_\perp^2(\zeta)d\zeta\right]^2}\right\},\end{aligned} \tag{22}$$

$$\begin{aligned}[R]\equiv{}&2\gamma^2\frac{\dot{\boldsymbol{\beta}}_\perp}{(z_0-z')\{1+\gamma^2[\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z')]^2\}^2}\\&\times([\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z')]\{1+\gamma^2[\boldsymbol{\beta}_\perp(z_0)-\boldsymbol{\beta}_\perp(z')]^2\\&-\gamma^2[\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z_0)]^2\}-[\boldsymbol{\beta}_\perp(z_0)-\boldsymbol{\beta}_\perp(z')]\\&\times\{1+\gamma^2[\boldsymbol{n}_\perp-\boldsymbol{\beta}_\perp(z')]^2\}).\end{aligned} \tag{23}$$

A rather straightforward calculation shows that the expression obtained is, indeed, regular in the limit $\Delta z\to 0$ or, equivalently, $(z_0-z')\to 0$. That is, it is sufficient to consider the case of constant transverse acceleration $\dot{\boldsymbol{\beta}}_\perp=$const. Without loss of generality, let us put $\dot{\beta}_x=\alpha$, $\dot{\beta}_y=0$. By shifting the origin and setting $z_0-z'\equiv\tau$, one has $\beta_y(z_0)=0$, $\beta_x(z_0)=\alpha\tau$. Upon this, the Coulomb part becomes

$$\begin{aligned}[C]={}&\frac{1+\frac{3}{4}\gamma^2\alpha^2\tau^2}{\tau^2\left(1+\frac{1}{4}\gamma^2\alpha^2\tau^2\right)^2}-\frac{1+\frac{1}{4}\gamma^2\alpha^2\tau^2}{\tau^2\left(1+\frac{1}{12}\gamma^2\alpha^2\tau^2\right)^2}\\\simeq{}&\frac{\gamma^2\alpha^2}{6}\frac{\left(1-\frac{1}{3}\gamma^2\alpha^2\tau^2\right)}{\left(1+\frac{1}{4}\gamma^2\alpha^2\tau^2\right)^2\left(1+\frac{1}{12}\gamma^2\alpha^2\tau^2\right)^2},\end{aligned} \tag{24}$$

which clearly has no pole as $\tau\to 0$. Similarly, one can check the absence of singularity in the radiative part.

IV. ENERGY LOSS FOR A TEST PARTICLE IN A ONE-DIMENSIONAL BUNCH

The next step is to evaluate the energy change for a test particle interacting with the whole bunch characterized by a given electron density distribution. This latter is, in accordance with our assumptions, stationary in a comoving frame of reference. It is also worth mentioning that, in terms of the 1D model used here, it is essentially the same (up to a numerical factor) as the longitudinal profile of the total current carried by the bunch.

As has already been said, we are interested only in the contribution coming from particles that are behind the test one; it is logical then to express the bunch density, which we will call λ, in terms of the longitudinal distance from the test particle. The corresponding variable Δz has already been introduced in Eq. (7). Then the energy change can be written as

$$\left(\frac{d\mathcal{E}}{dt}\right)_B(z_0)=\int_0^\infty\left(\frac{d\hat{\mathcal{E}}}{dt}\right)(z_0,\Delta z)\lambda(\Delta z)d(\Delta z), \tag{25}$$

where B stands for "*bunch*" and λ is supposed to vanish as $\Delta z\to+\infty$, so that the integral converges at the upper limit. Note that the lower limit of integration is zero.

Clearly, it is more convenient to perform integration over the retarded position z' rather than over the distance between particles Δz, since this eliminates the necessity of solving Eq. (11) for z'. Upon this, Eq. (25) becomes

$$\left(\frac{d\mathcal{E}}{dt}\right)_B(z_0)=\int_{z_0}^{-\infty}\left(\frac{d\hat{\mathcal{E}}}{dt}\right)(z_0,z')\lambda(\Delta z)\frac{d(\Delta z)}{dz'}dz', \tag{26}$$

where the limits of integration correspond to the retarded position of the source particle for $\Delta z=0$ or, respectively, $+\infty$. Note that $\lambda(\Delta z)$ is to be considered as a shorthand for $\lambda(\Delta z(z,z'))$. The expression for $(d\hat{\mathcal{E}}/dt)(z_0,z')$ was obtained in the previous section. As for $d(\Delta z)/dz'$, one can easily check that

$$\frac{d(\Delta z)}{dz'}=-[1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}(z')]. \tag{27}$$

TABLE I. Energy change in J for an electron located at the head of a bunch with rectangular density distribution. A comparison is made between evaluation with completely analytical formulas found by other authors and our simuation. B is the magnetic field in T, l_b is the bunch length in m, γ is the Lorentz factor, $\overline{z}$ is the length of the interaction zone in m, and N is the number of particles considered in the bunch.

Case	B (T)	l_b (m)	γ	$\overline{z}$ (m)	N (units of 10^9)	Analytical results (J)	Simulation results (J)
1	0.043	1.0×10^{-6}	25	1.2×10^{-2}	6.0	8.7×10^{-15}	8.3×10^{-15}
2	0.085	1.0×10^{-7}	50	8.0×10^{-3}	10.0	1.54×10^{-13}	1.50×10^{-13}
3	0.17	0.45	50	9.9×10^{-2}	10.0	3.7×10^{-17}	3.4×10^{-17}
4	0.85	0.2	500	0.02	10.0	8.4×10^{-17}	9.3×10^{-17}

Finally, if we want to obtain the energy loss during the entire trajectory we have to integrate over t (or, equivalently, over z_0), which gives

$$\Delta\mathcal{E}=\int_{-\infty}^{+\infty}\left(\frac{d\mathcal{E}}{dt}\right)_B\frac{dt}{dz_0}dz_0. \tag{28}$$

Equation (28) is a closed expression for the energy loss, in the sense that all we need to know is just the transverse velocity of the bunch as a function of the propagation distance. This latter is fully defined by the (predesigned) configuration of external magnetic fields. As a result, we get for the total energy change

$$\Delta\mathcal{E}\simeq\frac{e^2}{4\pi\epsilon_0 c}\int_{-\infty}^{+\infty}dz_0\int_{-\infty}^{z_0}dz'\{[C]+[R]\}\lambda(\Delta z), \tag{29}$$

where $[C]$ and $[R]$ are defined by Eqs. (22) and (23), and Δz by Eq. (11).

A useful particular case of the above equation is that of a rectangular current profile: $\lambda(\Delta z)$ is assumed to be constant, $\lambda(\Delta z)=\lambda_0$, over the whole length of the bunch l_b. If the test particle is situated at a distance s_0 from the head of the bunch, then the expression for the energy loss becomes

$$\Delta\mathcal{E}(s_0)\simeq\frac{e^2\lambda_0}{4\pi\epsilon_0 c}\int_{-\infty}^{+\infty}dz_0\int_{z'_*(l_b-s_0)}^{z_0}dz'\{[C]+[R]\}, \tag{30}$$

where $z'_*(l_b-s_0)$ stands for the solution of Eq. (11) corresponding to $\Delta z=l_b-s_0$, and s_0 is understood to be positive for particles that lie behind the head of the bunch.

We have performed a comparison of the above expressions with some earlier results obtained without the use of the small-angle approximation. Following [2], a general analysis of the problem of a bunch with rectangular density distribution passing through a bending magnet can be considerably simplified in several limiting cases. That is, the authors of [2] call the magnet *short* (*long*) if it deflects electrons at an angle much smaller (larger) than $1/\gamma$. On the other hand, an electron bunch is considered *short* or *long*, when its linear dimension is, respectively, much shorter or much longer than A/γ^3, where A is the radius of curvature of the particle trajectory in the magnet. From this, normalized expressions for the bunch length ($\hat{l}_b=l_b\gamma^3/A$) and for the angular dimension of the magnet ($\hat{\phi}_m=\gamma\phi_m$) are obtained.

In one of those limiting cases, the comparison is particularly simple: if the bunch is short and the bending magnet is, in the normalized sense, much longer than the bunch, then, as has been argued in [2], the transient effects at the interface between the straight path and the magnet can be neglected. This means that, in this particular case, we can assume all the retarded positions of the sources to lie within the bending magnet, and the situation becomes stationary.

For a rectangular bunch containing N particles, one finds upon a calculation similar to that in Eq. (24) that

$$\left(\frac{d\mathcal{E}}{dt}\right)_B=-\frac{1}{4\pi\varepsilon_0}\frac{4Ne^2\gamma c}{Al_b}\frac{\gamma u_s(8+\gamma^2u_s^2)}{(4+u_s^2\gamma^2)(12+\gamma^2u_s^2)}, \tag{31}$$

where

$$u_s\simeq\frac{2\gamma^2(l_b-s_0)}{A}. \tag{32}$$

One can easily check that Eqs. (31) and (32) coincide with those found in [2].

In general, the expressions are rather complicated, and the corresponding comparison can only be done numerically. A computer code has been developed and benchmarked against several limiting cases given in [2]. The results are presented in Table I. Cases 1 and 2 deal with a short bunch and a magnet longer than the bunch: here the crucial factor is the energy of the beam. The difference by a factor of 2 in the Lorentz factor is responsible for the increase by a factor of 16 in the energy change. In cases 3 and 4 the magnet is long and the bunch is much longer (again in the normalized sense) than the magnet; these two cases have been computed, respectively, with low- and high-energy bunches.

In all cases we observe a good agreement between our numerical computations and the corresponding analytical estimates (a relatively large discrepancy of the order of 10% in cases 3 and 4 is presumably a result of the logarithmic accuracy of the analytical expressions in [2]). It is also worth mentioning that in all four cases the total energy change is small as compared to the initial particle energy; specifically, the largest relative energy change of about 4% is found in case 2. This confirms consistency of the computational scheme, as was discussed in Sec. I.

As one more test, we have calculated the instantaneous power radiated by a particle located at the head of a bunch that enters into a bending magnet. This case demonstrates

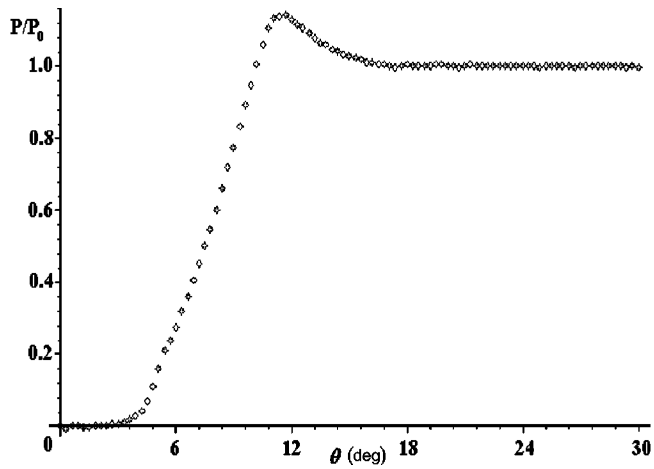

FIG. 3. Normalized transient power loss for a bunch with rectangular density distribution going into a bend.

pronounced transient collective effects. To be specific, we considered a 1 mm long, 40 MeV bunch with rectangular electron density distribution entering a circular trajectory with a radius $A=1$ m from a straight path. The dependence of the radiated power on the angle of deflection θ is shown in Fig. 3. After the bunch enters the bend the radiated power is seen to increase till it reaches a peak at $\theta \simeq 12°$. Upon this it decreases to its steady-state value, P_0, which has been used as an overall normalization factor in Fig. 3.

The observed dependence is in agreement with well-known results [3,4]. Basically, the transient in the figure connects two steady-state situations. The first one corresponds to the bunch before the bend: clearly, no power is radiated in this case. The second stationary regime is the steady-state CSR—that is, when the retarded positions of source particles interacting with the test particle are all in the bend. The transient describes a ''mixed'' situation when the retarded positions of source particles are partially in the bend and partially in the straight line preceding the bend.

V. CONCLUSIONS AND SPECULATIONS

An analytical approach to the problem of radiative collective effects within an ultrarelativistic electron bunch has been developed. The systematical use of the small-angle approximation results in a different expression for the energy exchange between a test particle and the bunch. This expression is closed, in the sense that we need to know only the transverse velocity of the bunch as a function of the propagation distance, which is directly determined by the external field configuration.

Analytical and numerical comparison of the formulas obtained with earlier results by other authors has been performed and good agreement has been demonstrated. The technique is applicable to an arbitrary bunch trajectory subject to only one restriction: a small deviation from the initial direction. A conceptual advantage of this route is that, due to the choice of geometry, we do not switch to a polar frame of reference, thus getting rid of any extra terms arising from the Jacobian of the transformation. We expect that, in future studies, this will allow us to account more easily for the finite transverse size of the bunch.

Note added in proof. Recently, discussions took place with Yaroslav Derbenev and Rui Li (both at Jefferson Lab) about the physical interpretation of noninertial space charge and centrifugal forces. Although these forces arise from the Jacobian transformation between Cartesian and cylindrical frames, they interpret them as existing independently of the choice of coordinate system.

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